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STUDENT VERSION
Modeling the Curved Path of a Bowling Ball

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Abstract: This modeling scenario examines the motion of a bowling ball with the goal of understanding how various aspects of the release affect ball placement and the entry angle to the pocket, each an important factor in the frequency of strikes. The equations of motion for a bowling ball are derived under modest simplifying assumptions, allowing students to examine the effect of initial velocity and spin on the path of the ball. The model can then be used to examine the sensitivity of various bowling strategies to small deviations in the initial conditions. Several ideas for additional study are included for students who wish to expand on the presented ideas.

SCENARIO DESCRIPTION

Bowling is a simple game. In each frame, a bowler has two opportunities to roll a spherical ball down a narrow lane with the goal of knocking down ten free standing pins. A game consists of ten such frames and the score of a game is based on the *pinfall*, i.e., the number of pins that are knocked down during each frame. Scoring is highly nonlinear – two bowlers who knock down the same number of pins can have very different scores. This nonlinearity stems from bonus points that are earned when a bowler knocks down all ten pins in any given frame. The bowler scores a *strike* by successfully knocking down all ten pins on the first attempt and earns bonus points equal to the total pinfall of the next two balls. The bowler scores a *spare* by successfully knocking down all ten pins using both attempts and earns bonus points equal to the total pinfall of the next ball. If necessary, an additional bonus frame (or two) will be provided to settle bonus scoring for the tenth frame. A *perfect game* consists of twelve consecutive strikes – the last two accounting for the bonus points of the tenth frame – and leads to 30 points per frame and a total score of 300. In contrast, a bowler who scores ten spares in one game can finish with a total score as low as 100. This disparity underscores the importance of strikes in competitive bowling.

The physical characteristics of the ball, pins, and lane play an important role in the difficulty of scoring a strike [3, 5]. The ball itself is characterized by its 27-inch circumference, while its weight typically ranges between 10 and 16 pounds for adult bowlers. The lane is 42 inches wide and the distance from the foul line (where the ball is released) to the center of the head pin, labeled as #1 in Figure 1, is 60 feet. Gutters on either side of the lane happily accept errant balls and such *gutter balls* generate no pinfall. The width of the lane is comprised of 39 narrow boards that run the length of the lane. Additionally, a set of ten dots are painted on the lane 6 feet from the foul line along with a series of arrows starting at a distance of approximately 12 feet from the foul line. These markings help bowlers plan and adjust their attempts from one ball to the next. The ten pins are arranged in a triangular pattern across four rows consisting of 1, 2, 3, and 4 pins, respectively. A scale illustration of a typical lane is shown in Figure **??** on the last page of this document. The shape and arrangement of the pins is more clearly illustrated by Figure 1, which includes an overhead view as well as the view from the bowler's perspective. Notice that pins 5, 8, and 9, respectively, stand directly behind pins 1, 2, and 3, which can lead to difficult spare attempts when these *sleeper* pins are obstructed after the first ball has been rolled.

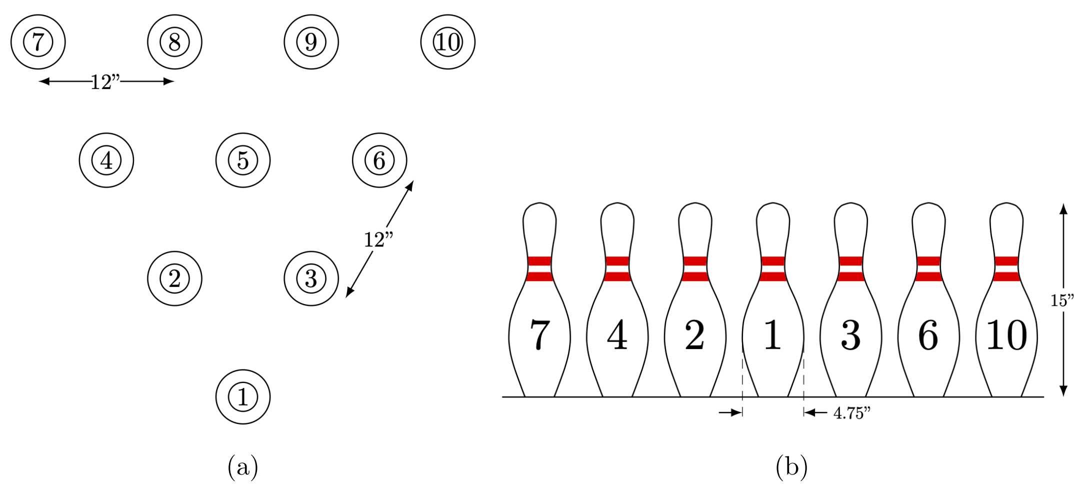


Figure 1. Pin arrangement and numbering convention.

The term *pocket* is used to refer to the gaps between either pins 1 and 2 or pins 1 and 3. Balls entering either pocket have the greatest chance to produce a strike. Testing performed by the United States Bowling Congress has shown that the angle of entry has a substantial effect on the frequency of strikes [4]. This research suggests that a six degree angle of entry produces the highest frequency of strikes and is less sensitive to small offset errors in ball placement within the pocket. As it turns out, achieving a six degree angle of entry is not so simple. The scale illustration of Figure **??** conveys the fact that a bowling lane is very narrow. In fact, a ball rolled from either side of the lane directly at the head pin will make an angle of

$$\theta = \tan^{-1}\left(\frac{1.75}{60}\right) \approx 1.67^\circ$$

and any ball following a linear path into the pocket must have a smaller angle of entry. The only way around this limitation is to apply spin to the ball so that it follows a curved path to the pocket. One other physical characteristic of the lane must now enter the discussion. Bowling lanes are heavily oiled and the oil pattern has a profound effect on the amount of *hook* that can be produced [2]. In this modeling scenario a typical *house pattern* will be adopted. House patterns place more oil towards the center of the lane and extend roughly forty feet from the foul line. A 2014 research paper by Banerjee and McPhee states that the coefficient of friction between the ball and the lane is approximately 0.04 in oiled areas, compared to 0.2 in dry areas [1]. In the absence of other information, these values will be used throughout this modeling scenario.

The goal of this modeling scenario is to develop differential equations that govern the path of a bowling ball in terms of the velocity and spin applied to the ball at the time it is released. These *equations of motion* can then be used to compare different strategies for placing the ball in the pocket with a satisfactory angle of entry.

Materials

This modeling scenario can be completed without any additional materials and, as written, makes use of no data. However, interested students are encouraged to visit a local bowling alley to develop intuition and possibly collect usable data.

- Position data may be extracted from a good video of a thrown ball.
- A stop watch could be used to estimate the speed of a thrown ball.
- A slow motion video may shed light on the spin rate of the ball.

1 Rolling with Slipping

The motion of the bowling ball will be modeled using the three-dimensional coordinate system illustrated in Figure 2. This coordinate system places the origin on the foul line at the right edge of the lane and respects the right-hand rule. The positive x-axis thus runs along the right-hand edge of the lane, while the positive y-axis runs along the foul line from right to left. The unit vectors in the direction of the x, y, and z axes will be denoted by $\vec{i}$, $\vec{j}$, and $\vec{k}$, respectively.

Consider a two-dimensional model of a bowling ball rolling with linear velocity v and angular velocity ω moving along the positive x-axis, as shown in Figure 3. In this illustration, v corresponds to the velocity of the center of the ball while ω represents the angular velocity (or spin rate) of the ball about its center, while R is the radius of the ball. Although both v and ω are assumed to vary with time (denoted by t), this dependence will be suppressed to simplify the notation.

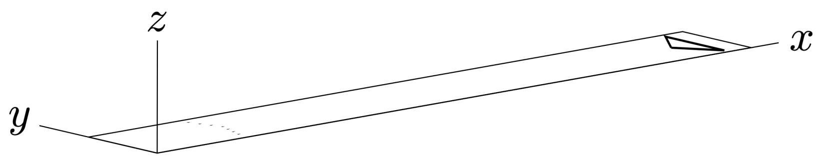


Figure 2. A three-dimensional coordinate system for the motion of a bowling ball.

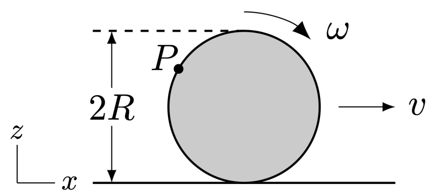


Figure 3. Two-dimensional illustration of a rolling bowling ball.

The velocity at a point P on the outer surface of the ball is a combination of the linear and angular velocities. The point on the top of the ball has velocity $v + \omega R$, while the point on the bottom of the ball has velocity $v - \omega R$. When $v = \omega R$, the velocity of the ball at the point of contact is zero and the ball rolls a distance equal to $2\pi R$ with each complete revolution of the ball. In this case, the ball is said to be in a state of *pure rolling* and will experience no acceleration. In contrast, when $v \neq \omega R$, the velocity of the ball at the point of contact is nonzero, which means the ball is *slipping.* If $v > \omega R$, the ball slips in the positive x direction at the point of contact and will be met with an opposing frictional force. This force will act to increase the spin rate of the ball while reducing the linear velocity of the ball. Similarly, if $v < \omega R$, the ball slips in the negative x direction at the point of contact and the frictional force will act in the positive x direction, decreasing the spin rate and increasing the linear velocity of the ball. A ball with excess spin ($v < \omega R$) will accelerate as it heads towards the pins, while a ball with a spin deficit ($v > \omega R$) will decelerate as it makes its way down the lane. A similar force acts in the y direction when a bowling ball is rotating about the x-axis, allowing the ball to follow a curved path as it heads down the lane.

It will be convenient to introduce vector notation for the development of a three-dimensional model of a rolling bowling ball. Let $\vec{v}$ represent the velocity vector of the center of the ball and let $\vec{v}_P$ represent the velocity at a point P on the surface of the ball. The *relative velocity* of P with respect to the center of the ball is defined as $\vec{v}_P - \vec{v}$ and can be computed in terms of the angular velocity of the ball and the position vector of P relative to the center of the ball, $\vec{r}_P$. Angular velocity will be represented as a vector $\vec{\omega}$ parallel to the axis of rotation with magnitude equal to the speed of rotation in radians per second. The direction of $\vec{\omega}$ is determined using the right-hand rule, i.e., when the fingers on one's right hand are curled about the axis of rotation in the direction of motion, the extended thumb will point in the direction of $\vec{\omega}$. The relative velocity $\vec{v}_P - \vec{v}$ must be perpendicular to both $\vec{\omega}$ and $\vec{r}_P$ with speed equal to the product of $\|\vec{\omega}\|$ and the perpendicular distance from P to the axis of rotation. This relationship can be described precisely using the cross-product as

$$\vec{v}_P = \vec{v} + \vec{\omega} \times \vec{r}_P. \tag{1}$$

As in the two-dimensional analysis, the velocity at the point of contact on the bottom of the ball is of primary interest. Let B represent the point of contact between the ball and the lane, so that $\vec{r}_B = -R\vec{k}$. Write $\vec{v} = v_x\vec{i} + v_y\vec{j}$ and $\vec{\omega} = \omega_x\vec{i} + \omega_y\vec{j} + \omega_z\vec{k}$, so that the ball may now have nonzero velocities in both the x and y directions while spinning about an arbitrary axis of rotation. Using (1), the velocity of the point on the bottom of the ball is given by

$$\begin{aligned} \vec{v}_B &= (v_x\vec{i} + v_y\vec{j}) + (\omega_x\vec{i} + \omega_y\vec{j} + \omega_z\vec{k}) \times (-R\vec{k}) \\ &= (v_x - \omega_y R)\vec{i} + (v_y + \omega_x R)\vec{j}. \end{aligned} \tag{2}$$

Notice that while it is certainly possible to spin a bowling ball around the z-axis, this component of $\vec{\omega}$ has no effect on $\vec{v}_B$ and thus will not influence the path of the ball. One possible configuration for a bowling ball at the time of release is illustrated in Figure 4. In this scenario, the ball would be released towards the left side of the foul line ($y = 2.75$ `ft`) and aimed slightly towards the right gutter ($v_x = 22$ `ft/s`, $v_y = -1.85$ `ft/s`) with some top spin ($\omega_y = 1$ `rev/s`) and a large amount of side spin ($\omega_x = -4.5$ `rev/s`). Notice that the ball is sliding forward and to the right, which means the frictional force will act to decrease v_x while increasing v_y. This will cause the ball to curve to the left as it rolls down the lane.

The magnitude of the frictional force acting on the ball at the point of contact with the lane is equal to μmg, where μ is the coefficient of kinetic friction, m is the mass of the ball, and g represents acceleration due to gravity. For the purposes of this modeling scenario, it will be assumed that $g \approx 32.17$ `ft/s`^{2}. The equations of motion for the ball can then be determined from the vector forms of Newton's Second Law,

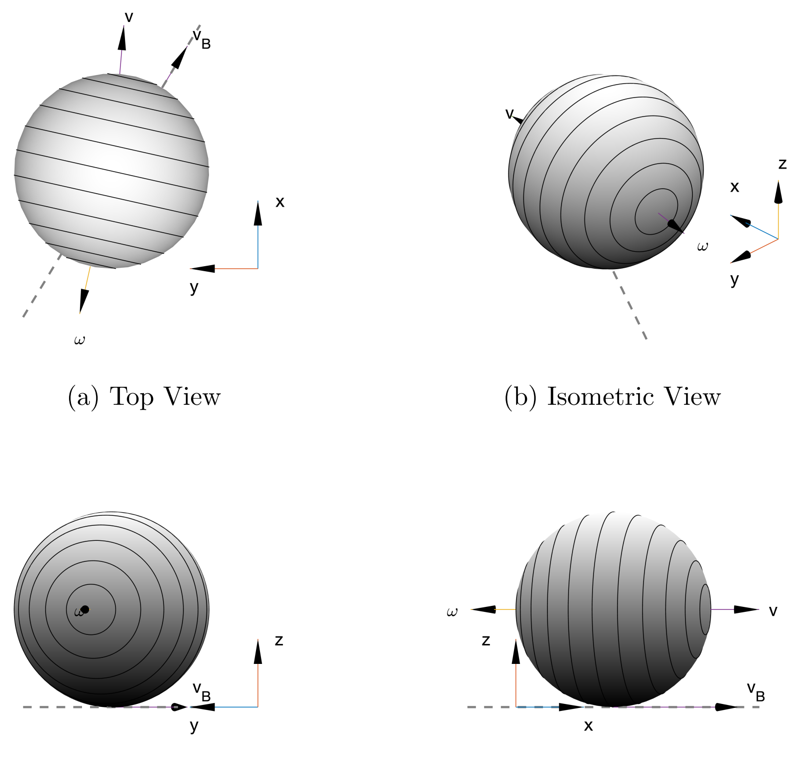
$$m\vec{a} = \vec{F} \qquad \text{and} \qquad I\vec{\alpha} = \vec{r}_B \times \vec{F}, \tag{3}$$

where $\vec{a}$ represents the acceleration vector of the center of the ball, $\vec{F}$ is the frictional force at point B, I is the rotational inertia of the ball about its center, and $\vec{\alpha}$ is the angular acceleration of the ball about its center. For simplicity, the equations of motion will be derived under the assumption that $\vec{v}_B \neq \vec{0}$, but the reader should note that both $\vec{a}$ and $\vec{\alpha}$ are zero when $\vec{v}_B = \vec{0}$, i.e., once the ball achieves a state of pure rolling. Recall that the frictional force $\vec{F}$ acts in the opposite direction to $\vec{v}_B$, so that

$$\vec{F} = -\mu mg \frac{\vec{v}_B}{\|\vec{v}_B\|}. \tag{4}$$

Recall that for any vector $\vec{v}$: (1) $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$ and (2) $\vec{v}/\|\vec{v}\|$ is a unit vector in the same direction as $\vec{v}$. These facts will be used again below. Notice that (4) makes it possible to eliminate the mass m from (3), leading to the following expression for the acceleration of the ball,

$$\vec{a} = -\mu g \frac{\vec{v}_B}{\|\vec{v}_B\|}. \tag{5}$$



(a) Top View

(b) Isometric View

(c) Bowler's View

(d) Side View

Figure 4. An illustration of a bowling ball rolling with slipping.

Modeling the bowling ball as a uniform sphere, its rotational inertia is given by $I = \frac{2}{5}mR^2$. The angular acceleration can then be written as

$$\begin{aligned} \vec{\alpha} &= -\frac{5\mu g}{2R^2}\left(\vec{r}_B \times \frac{\vec{v_B}}{\|\vec{v}_B\|}\right) \\ &= \frac{5\mu g}{2R}(\vec{k} \times \left(\frac{(v_B)_x}{\|v_B\|}\vec{i} + \frac{(v_B)_y}{\|v_B\|}\vec{j}\right) \\ &= \frac{5\mu g}{2R}\left(-\frac{(v_B)_y}{\|v_B\|}\vec{i} + \frac{(v_B)_x}{\|v_B\|}\vec{j}\right). \end{aligned} \tag{6}$$

Recall that $\vec{a} = \frac{d\vec{v}}{dt}$ and $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$. Combining these identities with the observation that $\frac{dx}{dt} = v_x$ and $\frac{dy}{dt} = v_y$, one obtains a system of six first-order differential equations for the motion of a bowling ball:

$$\begin{aligned} \frac{dx}{dt} &= v_x & \frac{dy}{dt} &= v_y \\ \frac{dv_x}{dt} &= -\mu g\frac{(v_B)_x}{\|\vec{v}_B\|} & \frac{dv_y}{dt} &= -\mu g\frac{(v_B)_y}{\|\vec{v}_B\|} \\ \frac{d\omega_x}{dt} &= -\frac{5\mu g}{2R}\frac{(v_B)_y}{\|\vec{v}_B\|} & \frac{d\omega_y}{dt} &= \frac{5\mu g}{2R}\frac{(v_B)_x}{\|\vec{v}_B\|}. \end{aligned} \tag{7}$$

These differential equations apply so long as the ball is rolling with slipping. Once the ball reaches a state of pure rolling, it will continue with constant velocities in both the x and y directions.

Questions:

1. The following exercises explore some subtle aspects of rolling with slipping for a bowling ball.

   a. Use the equations of motion (7) to create a differential equation for $\vec{v}_B$.

   b. Use the differential equation from (a) to explain the fact that the direction of the frictional force on a bowling ball in motion is constant up to the moment that pure rolling is achieved. Is the magnitude of the frictional force also constant? Explain your answer.

   c. Assume that μ is constant. Derive a formula in terms of the initial conditions for the time when a bowling ball will reach a state of pure rolling.

2. The goal of this problem is to determine the path of a bowling ball from any given initial conditions.

   a. Assume that μ is constant. Solve the equations of motion (7) to obtain explicit solutions for $x(t)$ and $y(t)$ both before and after the moment that the ball achieves a state of pure rolling.

   b. Explain how one can use the solution from (a) to account for the transition from the oiled section of the lane to the dry section of the lane.

   c. Find expressions for $x(t)$ and $y(t)$ using the initial conditions of Table 1 under the assumption that $\mu = 0.04$ for $0 \leq x \leq 40$ and $\mu = 0.20$ for $x > 40$. Graph the path of the ball and describe any interesting features.

Table 1. Initial conditions for Question 2(c).

$x(0)$ ft	$y(0)$ ft	$v_x(0)$ ft/s	$v_y(0)$ ft/s	$\omega_x(0)$ rev/s	$\omega_y(0)$ rev/s
0	0.8	20	0	-1	2

3. Recall that the greatest frequency of strikes occurs when the angle of entry to the 1–3 pocket is approximately 6 degrees. Adjust the initial conditions of the model to create a strategy for achieving a near-optimal angle of entry. Are the chosen initial conditions physically reasonable?

4. Notice that the mass m of the bowling ball does not appear in the equations of motion (7). Does the mass of the ball play any role in a bowler's ability to achieve a desired path or entry angle? Explain your answer.

Ideas for Further Exploration

A. Sensitivity Study: Choose one or more bowling styles and use the mathematical model to identify initial conditions that produce a satisfactory angle of entry. Examine the effect of small deviations in the initial conditions on the ball placement and angle of entry. It might be interesting to hypothesize about how a bowler's game changes when they are tired.

B. Spin Rate: Describe an approach to calculating the number of rotations a ball makes on its way down the lane. Compare the number of rotations made on the oiled and dry sections of the lane. What is the smallest number of rotations that can reasonably be achieved? Can a slow motion video provide information about typical spin rates of a bowling ball?

C. Strategy for Spares: Formulate and test a hypothesis about whether or not a "hook shot" increases the margin of error when attempting a spare. The answer may depend on the specific layout of the pins as well as the type of hook shot used. Use the mathematical model to measure the effect of error in the release angle on the desired ball placement.

D. Oil Patterns: This modeling scenario assumes that oil is distributed uniformly in the oiled section of the lane. Modern bowling alleys uses programmable units that are capable of applying a wide variety of oil patterns for competitive bowling. One could attempt to capture greater realism by implementing a two-dimensional model for the coefficient of friction, e.g., increasing the coefficient of friction near the gutters.

E. Asymmetric Balls: Many bowling balls include an asymmetric core that alters the rotational inertia of the ball. The core is designed to increase the hook of the ball when rolled about a specific axis. Adapt the model to account for such a design and test its effect on the achievable angle of entry.

F. Pin Action: Rotation of the bowling ball about the z axis has no effect on the path followed by the ball since this component of rotation does not contribute to $\vec{v}_B$, the relative velocity of the point on the bottom of the ball. Nevertheless, it is natural to wonder if a high spin rate about the z axis has any effect on the movement of the pins once the ball makes contact.

G. Oil Absorption: A variety of materials are used for the outer shell of modern bowling balls. Some bowling balls are designed to absorb oil in order to achieve a higher coefficient of friction and greater hook. In fact, most bowling alleys now have coin-operated machines that are designed to clean a ball, removing any absorbed oil. It could be interesting to examine the long term behavior of an absorptive bowling ball that is not cleaned.

Acknowledgement

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